Homotopy type theory

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Sections 1–3 have been adapted quite directly from Andrej Bauer's Coq scripts, available at `https://github.com/andrejbauer/Homotopy/tree/master/UnivalentFoundations`. Section 4 consists of a proof by Mike Shulman, and original observations.

1 Basics

`existT` **Definition 1.1.** A *fibration* over A is $\Sigma x : A.P(x)$. The *fiber of* $x : A$ is the type $P(x)$.

`paths` **Definition 1.2.** For any $x, y : A$ we have the type of *paths in* A between them, $x \rightsquigarrow y$. The only constructor is $\mathsf{id}_x : x \rightsquigarrow x$.

`paths_rect` *Remark* 1.1. The elimination (J) rule states that, given a fibration P over paths in A in which the fiber of each id_x is inhabited, then the fiber of any path is inhabited.

Remark 1.2. Univalence (Axiom 3.1) introduces additional path constructors between types.

`concat` **Proposition 1.1.** *Given $p : x \rightsquigarrow y$ and $q : y \rightsquigarrow z$, we can construct $pq : x \rightsquigarrow z$.*

Proof. Consider the fibration $P(a \rightsquigarrow b) = (x \rightsquigarrow a) \to (x \rightsquigarrow b)$. Each fiber $P(\mathsf{id}_a) = (x \rightsquigarrow a) \to (x \rightsquigarrow a)$ is inhabited by the identity function, so by J, we have a term of type $P(y \overset{q}{\rightsquigarrow} z) = (x \rightsquigarrow y) \to (x \rightsquigarrow z)$. Applying this to p yields the desired path $x \rightsquigarrow z$. □

`opposite` **Proposition 1.2.** *Given $p : x \rightsquigarrow y$, we can construct $p^{-1} : y \rightsquigarrow x$.*

Proof. Consider the fibration $P(a \rightsquigarrow b) = (a \rightsquigarrow x) \to (b \rightsquigarrow x)$. Each fiber $P(\mathsf{id}_a)$ is inhabited by the identity function, so by J, we have a term of type $P(x \overset{p}{\rightsquigarrow} y) = (x \rightsquigarrow x) \to (y \rightsquigarrow x)$. Applying this to id_x yields the desired path $y \rightsquigarrow x$. □

We are often interested in paths between paths, which we call *homotopies*. The following propositions concern homotopies, and can be proven with an appropriate choice of fibration.

Proposition 1.3. *Assuming $x, y, z : A$, $p : x \rightsquigarrow y$, and $q : y \rightsquigarrow z$, the following homotopies are inhabited:*

`idpath_left_unit` 1. $\mathsf{id}_x p \rightsquigarrow p$,

`idpath_right_unit` 2. $p\,\mathsf{id}_y \rightsquigarrow p$,

3. $\mathsf{id}_x^{-1} \rightsquigarrow \mathsf{id}_x$,

`pposite_right_inverse` 4. $pp^{-1} \rightsquigarrow \mathsf{id}_x$,

`opposite_left_inverse` 5. $p^{-1}p \rightsquigarrow \mathsf{id}_y$,

`opposite_concat` 6. $(pq)^{-1} \rightsquigarrow q^{-1}p^{-1}$, *and*

`opposite_opposite` 7. $(p^{-1})^{-1} \rightsquigarrow p$.

`concat_associativity` **Proposition 1.4.** *Given* $w \overset{p}{\rightsquigarrow} x$, $x \overset{q}{\rightsquigarrow} y$, *and* $y \overset{r}{\rightsquigarrow} z$, $(pq)r \rightsquigarrow p(qr)$.

`homotopy_concat` **Proposition 1.5.** *Given* $p, p' : x \rightsquigarrow y$, $q, q' : y \rightsquigarrow z$, $h_p : p \rightsquigarrow p'$, *and* $h_q : q \rightsquigarrow q'$, $pq \rightsquigarrow p'q'$ *is inhabited.*

`map` **Definition 1.3.** Given $f : A \to B$, we can construct the map $f_* : \Pi x\ y : A.(x \rightsquigarrow y) \to (f\ x \rightsquigarrow f\ y)$.

Proposition 1.6. $f_*(p)$ *is functorial in both* f *and* p, *that is,*

`idpath_map` 1. $f_*(\mathsf{id}_x) \rightsquigarrow \mathsf{id}_{f(x)}$,

`concat_map` 2. $f_*(pq) \rightsquigarrow f_*(p)f_*(q)$,

`idmap_map` 3. $(\lambda x.x)_*(p) \rightsquigarrow p$, *and*

`composition_map` 4. $(g \circ f)_*(p) \rightsquigarrow (g_* \circ f_*)(p)$.

Furthermore,

`opposite_map` 5. $f_*(p^{-1}) \rightsquigarrow (f_*(p))^{-1}$ *and*

`map_cancel` 6. $p \rightsquigarrow q$ *implies* $f_*(p) \rightsquigarrow f_*(q)$.

`homotopy_natural` **Proposition 1.7.** *Given maps* $f, g : A \to B$, *a family of paths* $p_x : f(x) \rightsquigarrow g(x)$ *in* B, *and a path* $q : x \rightsquigarrow y$, *the following diagram of paths commutes up to homotopy.*

$$\begin{tikzcd} f(x) \arrow[r, "p_x"] \arrow[d, "f_* q"'] & g(x) \arrow[d, "g_* q"] \\ f(y) \arrow[r, "p_y"] & g(y) \end{tikzcd}$$

Proof. Consider the fibration $P(a \overset{r}{\rightsquigarrow} b) = f_*(r)p_b \rightsquigarrow p_a g_*(r)$. If each $P(\mathsf{id}_a)$ is inhabited, then $P(x \overset{q}{\rightsquigarrow} y) = f_*(q)p_y \rightsquigarrow p_x g_*(q)$ is inhabited, which is exactly our goal. So it suffices to show $P(\mathsf{id}_a)$ is inhabited.

$P(\mathsf{id}_a) = f_*(\mathsf{id}_a)p_a \rightsquigarrow p_a g_*(\mathsf{id}_a)$. Because paths can be concatenated, it suffices to exhibit paths $f_*(\mathsf{id}_a)p_a \rightsquigarrow p_a$ and $p_a \rightsquigarrow p_a g_*(\mathsf{id}_a)$. For the former, $f_*(\mathsf{id}_a) \rightsquigarrow \mathsf{id}_{f(a)}$ by functoriality, so $f_*(\mathsf{id}_a)p_a \rightsquigarrow \mathsf{id}_{f(a)}p_a$, and $\mathsf{id}_{f(a)}p_a \rightsquigarrow p_a$ by left identity; by concatenation, $f_*(\mathsf{id}_a)p_a \rightsquigarrow p_a$. The latter path is constructed similarly. □

Proposition 1.8. *Let* p, q, r *be paths in* A.

`concat_cancel_right` 1. *If* $p, q : x \rightsquigarrow y$, $r : y \rightsquigarrow z$, *and* $h : pr \rightsquigarrow qr$, *then* $p \rightsquigarrow q$.

`concat_cancel_left` 2. *If $p : x \rightsquigarrow y$, $q, r : y \rightsquigarrow z$, and $h : pq \rightsquigarrow pr$, then $q \rightsquigarrow r$.*

Proof. We prove only (1); (2) is analogous. By Proposition 1.5 at the homotopies $h : pr \rightsquigarrow qr$ and $\mathsf{id}_{r^{-1}} : r^{-1} \rightsquigarrow r^{-1}$, we get a path $prr^{-1} \rightsquigarrow qrr^{-1}$. Applying this again with $p \rightsquigarrow p$ and $rr^{-1} \rightsquigarrow \mathsf{id}_y$, $prr^{-1} \rightsquigarrow p\mathsf{id}_y$, and by right identity, $p\mathsf{id}_y \rightsquigarrow p$. Similarly, $qrr^{-1} \rightsquigarrow q$, so by concatenation and inversion, $p \rightsquigarrow q$. □

`concat_move_over_left` **Proposition 1.9.** *If $p : x \rightsquigarrow z$, $q : x \rightsquigarrow y$, $r : y \rightsquigarrow z$, and $p \rightsquigarrow qr$, then $pr^{-1} \rightsquigarrow q$.*

Proof. By right cancellation, it suffices to show $pr^{-1}r \rightsquigarrow qr$. $r^{-1}r \rightsquigarrow \mathsf{id}_z$, so $pr^{-1}r \rightsquigarrow p\mathsf{id}_z$, $p\mathsf{id}_z \rightsquigarrow p$ by right identity, and $p \rightsquigarrow qr$ by hypothesis. □

`omap_homotopy_commute` **Proposition 1.10.** *Given $f : A \to A$ and a family of paths $p_x : f(x) \rightsquigarrow x$, then $f_*(p_x) \rightsquigarrow p_{f(x)}$.*

Proof. It suffices to exhibit paths $f_*(p_x) \rightsquigarrow f_*(p_x)p_xp_x^{-1} \rightsquigarrow p_{f(x)}$. For the former, $f_*(p_x) \rightsquigarrow f_*(p_x)\mathsf{id}_{f(x)} \rightsquigarrow f_*(p_x)p_xp_x^{-1}$. For the latter, by Proposition 1.9, it suffices to show $f_*(p_x)p_x \rightsquigarrow p_{f(x)}p_x$. But by Proposition 1.7 at f and $(\lambda a : A.a)$,

$$
\begin{tikzcd}
f(f(x)) \arrow[r, "p_{f(x)}"] \arrow[d, "f_* p_x"'] & f(x) \arrow[d, "(\lambda a.a)_* p_x"] \\
f(x) \arrow[r, "p_x"] & x
\end{tikzcd}
$$

This yields the desired path by the additional homotopy $(\lambda a.a)_* p_x \rightsquigarrow p_x$. □

`map_action` **Proposition 1.11.** *Given $f : A \to A$, a family of paths $p_x : f(x) \rightsquigarrow x$, and $q : f(z) \rightsquigarrow y$, the following diagram of paths commutes up to homotopy.*

$$
\begin{tikzcd}
f(f(z)) \arrow[r, "f_* p_z"] \arrow[d, "f_* q"'] & f(z) \arrow[d, "q"] \\
f(y) \arrow[r, "p_y"'] & y
\end{tikzcd}
$$

Proof. Exhibit paths $f_*(p_z)q \rightsquigarrow p_{f(z)}q \rightsquigarrow f_*(q)p_y$. The former follows from Proposition 1.10, which yields a path $f_*(p_z) \rightsquigarrow p_{f(z)}$. The latter follows directly from Proposition 1.7 at f and $(\lambda a.a)$. □

2 Homotopies and equivalences

Note that we have two notions of homotopies between maps: given maps $f, g : A \to B$, we can have a homotopy $f \rightsquigarrow g$, or a pointwise homotopy $\Pi x : A.f(x) \rightsquigarrow g(x)$. It is clear that a homotopy gives rise to a pointwise homotopy.

`happly` **Proposition 2.1.** *Given $f, g : A \to B$ and $f \rightsquigarrow g$, then for each x, $f(x) \rightsquigarrow g(x)$.*

Proof. Consider the fibration $P(u \rightsquigarrow v) = \Pi x : A.u(x) \rightsquigarrow v(x)$ over paths between functions $A \to B$. $P(\mathsf{id}_u) = \Pi x : A.u(x) \rightsquigarrow u(x)$, which is inhabited at each x by $\mathsf{id}_{u(x)}$; thus $P(f \rightsquigarrow g) = \Pi x : A.f(x) \rightsquigarrow g(x)$ is inhabited. □

But here a pointwise homotopy must depend continuously on the choice of x, so a pointwise homotopy also gives rise to a homotopy; this statement amounts to *function extensionality* (Theorem 3.2), and requires univalence (Axiom 3.1).

`transport` **Theorem 2.1** (Transport)**.** *Given a fibration P over A and a path $p : x \rightsquigarrow y$ in A, we can* transport *an element of $P(x)$* over *p to an element $\tau_p(x)$ of $P(y)$.*

Proof. Consider the fibration $P'(a \rightsquigarrow b) = P(a) \to P(b)$. Each $P'(\mathsf{id}_a)$ is inhabited by the identity function. Given $x' : P(x)$, a term of type $P'(p)$ applied to x' yields an element of $P(y)$, which we call $\tau_p(x)$. □

Remark 2.1. Note that $\tau_{\mathsf{id}_x}(x') = x'$ by our definition of P'.

`total_paths` **Proposition 2.2.** *Given a fibration P over A with elements $x' : P(x)$ and $y' : P(y)$, a path $p : x \rightsquigarrow y$ in A, and a path $p' : \tau_p(x') \rightsquigarrow y'$ in $P(y)$, we can construct a path $(x, x') \rightsquigarrow (y, y')$ in P.*

Proof. Consider the fibration $P'(a \overset{p}{\rightsquigarrow} b) = \Pi a' : P(a).\Pi b' : P(b).(\tau_p(a') \rightsquigarrow b') \to ((a, a') \rightsquigarrow (b, b'))$. Clearly, a term of type $P'(p)$ applied to x', y', p' yields the desired path; hence it suffices to show $P'(\mathsf{id}_a)$ is inhabited, that is, given $a', a'' : P(a)$ and $q : \tau_{\mathsf{id}_a}(a') \rightsquigarrow a''$, there is a path $(a, a') \rightsquigarrow (a, a'')$. Since these are in the same fiber, we need only a path $a' \rightsquigarrow a''$. But $\tau_{\mathsf{id}_a}(a') = a'$, so q suffices. □

`base_path` **Proposition 2.3.** *Conversely, given a fibration P over A and a path $p : (x, x') \rightsquigarrow (y, y')$ in P, we can construct a path $q : x \rightsquigarrow y$ in A and a path $q' : \tau_p(x') \rightsquigarrow y'$ in P.*

Proof. Consider the fibration $P'((a, a') \rightsquigarrow (b, b')) = \Sigma p : a \rightsquigarrow b.\tau_p(a') \rightsquigarrow b'$. $P'(\mathsf{id}_{(a,a')})$ is inhabited by $(\mathsf{id}_a, \mathsf{id}_{a'})$, since $\tau_{\mathsf{id}_a}(a') = a'$. Then we have a term of type $P'(p)$, which is precisely the desired (q, q'). □

Remark 2.2. By the previous two propositions, $(x, x') \rightsquigarrow (y, y')$ in $\Sigma x : A.P(x)$ is inhabited if and only if $\Sigma p : x \rightsquigarrow y.\tau_p(x') \rightsquigarrow y'$ is inhabited.

`contractible` **Definition 2.1.** A space A is *contractible* if $\Sigma x : A.\Pi y : A.y \rightsquigarrow x$ is inhabited, i.e., there is a point $x : A$ to which every point $y : A$ has a path.

`hfiber` **Definition 2.2.** A *homotopy fiber* for $f : A \to B$ is a fiber of $\Sigma y : B.\Sigma x : A.f(x) \rightsquigarrow y$.

`transport_hfiber` **Proposition 2.4.** *Given an element (x, q) of the homotopy fiber $P(z) = \Sigma x : A.f(x) \rightsquigarrow z$ of $f : A \to B$, we can transport $q : P(z)(x)$ over $p : x \rightsquigarrow y$ to $\tau_p(q) \rightsquigarrow f_*(p^{-1})q$ in $P(z)(y)$.*

Proof. Consider the fibration $P'(a \overset{p}{\rightsquigarrow} b) = \Pi q : f(a) \rightsquigarrow z.\tau_p(q) \rightsquigarrow f_*(p^{-1})q$. Applying a term of type $P'(p)$ to q yields the desired path. It suffices to show $P'(\mathsf{id}_a) = \tau_{\mathsf{id}_a}(q) \rightsquigarrow f_*(\mathsf{id}_a^{-1})q$ is inhabited. By the definition of τ, $\tau_{\mathsf{id}_a}(q) = q$. $f_*(\mathsf{id}_a^{-1}) \rightsquigarrow f_*(\mathsf{id}_a) \rightsquigarrow \mathsf{id}_{f(a)}$, so $f_*(\mathsf{id}_a^{-1})q \rightsquigarrow \mathsf{id}_{f(a)}q \rightsquigarrow q$. □

`is_equiv` **Definition 2.3.** A map $f : A \to B$ is an *equivalence* if its homotopy fibers are contractible.

`idequiv` **Proposition 2.5.** *The identity function on A is an equivalence.*

Proof. We must show that, for any $y : A$, the homotopy fiber $P(y) = \Sigma x : A.x \rightsquigarrow y$ is contractible. In particular, to contract the fiber to the point (y, id_y) in $P(y)$, we need a path $(x, p) \rightsquigarrow (y, \mathsf{id}_y)$ for every $(x, x \overset{p}{\rightsquigarrow} y)$ in $P(y)$. By Proposition 2.2, we need only show $x \rightsquigarrow y$ and $\tau_p(p) \rightsquigarrow \mathsf{id}_y$. The former is given by p; for the latter, by Proposition 2.4, $\tau_p(p) \rightsquigarrow (\lambda x.x)_*(p^{-1})p$, but $(\lambda x.x)_*(p^{-1}) \rightsquigarrow p^{-1}$, so $\tau_p(p) \rightsquigarrow p^{-1}p \rightsquigarrow \mathsf{id}_y$. □

path_to_equiv

Proposition 2.6. *Given $p : A \rightsquigarrow B$, there is an equivalence $A \to B$.*

Proof. Consider the fibration P such that $P(A \rightsquigarrow B)$ is the type of equivalences $A \to B$. By Proposition 2.5, $P(\mathsf{id}_A)$ is inhabited by the identity function, so we have a term of type $P(p)$, which is the desired equivalence. □

equiv_inv

equiv_inv_is_*

Proposition 2.7. *Given an equivalence $f : A \to B$, we obtain a map $f^{-1} : B \to A$ such that $f(f^{-1}(b)) \rightsquigarrow b$ and $f^{-1}(f(a)) \rightsquigarrow a$.*

Proof. First, we construct f^{-1}. Given $b : B$, we need an element of A. Because f is an equivalence, the homotopy fiber of b, $\Sigma x : A.f(x) \rightsquigarrow b$, is contractible; in particular, the fiber is equipped with a center of retraction $(a, f(a) \overset{p}{\rightsquigarrow} b)$. Hence we have an element a of A.

It remains to show that f^{-1} is indeed an inverse of f up to homotopy. By the center of retraction above, we have a path $p : f(a) \rightsquigarrow b$. Since $f^{-1}(b) = a$ by definition, $p : f(f^{-1}(b)) \rightsquigarrow b$.

Finally, we show $f^{-1}(f(a)) \rightsquigarrow a$. The homotopy fiber of $f(a)$, $\Sigma x : A.f(x) \rightsquigarrow f(a)$, has a center of retraction $(x, f(x) \overset{p}{\rightsquigarrow} f(a))$, where $f^{-1}(f(a)) = x$ by definition. Because $(a, \mathsf{id}_{f(a)})$ is in the same fiber, the retraction yields a path $(x, p) \rightsquigarrow (a, \mathsf{id}_{f(a)})$. In particular, by Proposition 2.3, we have a path $x \rightsquigarrow a$, which completes the proof. □

equiv_injective

Proposition 2.8. *Given an equivalence $f : A \to B$ and a path $p : f(x) \rightsquigarrow f(y)$, then $x \rightsquigarrow y$.*

Proof. $(f^{-1})_*(p) : f^{-1}(f(x)) \rightsquigarrow f^{-1}(f(y))$. But by Proposition 2.7, $f^{-1}(f(x)) \rightsquigarrow x$ and $f^{-1}(f(y)) \rightsquigarrow y$, so $x \rightsquigarrow y$. □

3 Univalence and extensionality

All results in this section assume the eta and univalence axioms.

univalence_statement

Axiom 3.1 (Univalence)**.** The space $A \rightsquigarrow B$ and the space of equivalences $A \to B$ are equivalent, via the construction in Proposition 2.6.

eta_axiom

Axiom 3.2 (Eta)**.** For any $f : A \to B$, $(\lambda x.f\ x) \rightsquigarrow f$.

equiv_to_path

Proposition 3.1. *Given an equivalence $A \to B$, there is a path $A \rightsquigarrow B$.*

Proof. By univalence, there is an equivalence between the path space $A \rightsquigarrow B$ and the space of equivalences $A \to B$. By Proposition 2.7, we can invert this equivalence to yield a map from equivalences $A \to B$ to paths $A \rightsquigarrow B$, which is exactly what we need. □

Remark 3.1. This result is the essence of univalence: it posits a (non-identity) path between any equivalent spaces.

equiv_to_path_section

Proposition 3.2. *Given an equivalence $f : A \to B$, we may construct by Proposition 3.1 a path $p : A \rightsquigarrow B$, which by Proposition 2.6 yields an equivalence $f' : A \to B$. Then $f' \rightsquigarrow f$.*

Proof. By univalence, there is an equivalence g from $A \rightsquigarrow B$ to equivalences $A \to B$. Then $f' = g^{-1}(g(f))$ by the constructions of Propositions 2.6 and 3.1. By Proposition 2.7, $g^{-1}(g(f)) \rightsquigarrow f$, which is the desired path. $\square$

`pred_equiv_to_path` **Proposition 3.3.** *Given a fibration P over equivalences $A \to B$, we can construct a fibration P' over paths $A \rightsquigarrow B$.*

Proof. Define $P'(A \overset{p}{\rightsquigarrow} B)$ to be the fiber in P of the equivalence obtained from p by Proposition 2.6. $\square$

`equiv_induction` **Theorem 3.1** (Equivalence induction)**.** *Given a fibration P over equivalences $A \to B$, if the fiber of each identity map is inhabited, then the fiber of any equivalence is inhabited.*

Proof. Consider the fibration P' over paths $A \rightsquigarrow B$ obtained from P via Proposition 3.3. The fiber of each identity path is inhabited, because Proposition 2.6 sends identity paths to identity maps.

Given an equivalence $f : A \to B$, by Proposition 3.1 we get a path $p : A \rightsquigarrow B$. Then $P'(p)$ is inhabited by induction, but by construction, $P'(p) = P(f')$, where f' is the equivalence obtained from p. By Proposition 3.2, $f' \rightsquigarrow f$, so by transport, $P(f)$ is inhabited. $\square$

`equiv_pointwise_idmap` **Proposition 3.4.** *For any map $f : A \to A$, if $\Pi x : A.f(x) \rightsquigarrow x$, then f is an equivalence.*

Proof. Let $P = \Pi x : A.f(x) \rightsquigarrow x$. To show f is an equivalence, we show that the homotopy fiber of each $x : A$, $\Sigma y : A.f(y) \rightsquigarrow x$, is contractible with center $(x, P(x))$; that is, $(y, f(y) \overset{p}{\rightsquigarrow} x) \rightsquigarrow (x, f(x) \overset{P(x)}{\rightsquigarrow} x)$ for any y. By Proposition 2.2 and $y \overset{P(y)^{-1}}{\rightsquigarrow} f(y) \overset{p}{\rightsquigarrow} x$, we need only show a path $\tau_{P(y)^{-1}p}(p) \rightsquigarrow P(x)$. By Proposition 2.4, $\tau_{P(y)^{-1}p}(p) \rightsquigarrow f_*((P(y)^{-1}p)^{-1})p$, so it suffices to show $f_*((P(y)^{-1}p)^{-1})p \rightsquigarrow P(x)$:

$$\begin{aligned}
f_*((P(y)^{-1}p)^{-1})p &\rightsquigarrow f_*(p^{-1}P(y))p \\
&\rightsquigarrow f_*(p^{-1})f_*(P(y))p \\
&\rightsquigarrow f_*(p^{-1})f_*(p)P(x) && \text{by Proposition 1.11} \\
&\rightsquigarrow f_*(\mathsf{id}_x)P(x) \\
&\rightsquigarrow P(x).
\end{aligned}$$

$\square$

`etaequiv` **Proposition 3.5.** *For any $f : A \to B$, $f \mapsto (\lambda x.f\ x)$ is an equivalence from $A \to B$ to $A \to B$.*

Proof. The result follows directly from Proposition 3.4 and the eta axiom. $\square$

`equiv_exponential` **Proposition 3.6.** *For any equivalence $f : A \to B$, $(f \circ -)$ is an equivalence $(C \to A) \to (C \to B)$.*

Proof. Consider the fibration mapping an equivalence $g : A \to B$ to a proof that $(g \circ -)$ is an equivalence. To show $f \circ -$ is an equivalence, by equivalence induction it suffices to show that $\mathsf{id}_A \circ -$ is an equivalence. But $\mathsf{id}_A \circ - = \lambda f.\lambda x.\mathsf{id}_A(f\ x) = \lambda f.\lambda x.f\ x$, which is an equivalence by Proposition 3.5. $\square$

`path_space` **Definition 3.1.** For any type A, we have the *path space* of A, $Path(A) = \Sigma(x, y) : A \times A.x \rightsquigarrow y$.

`src,trg` **Proposition 3.7.** *The projections $p_0, p_1 : Path(A) \to A$ are equivalences.*

Proof. We show that the homotopy fiber of each $x : A$, $\Sigma((x_0, x_1), x_0 \overset{p}{\rightsquigarrow} x_1) : Path(A).x_i \rightsquigarrow x$, is contractible with center of retraction $(((x, x), \mathsf{id}_x), \mathsf{id}_x)$. That is, given any $a_0, a_1 : A$, $q : a_0 \rightsquigarrow a_1$, and $r : a_i \rightsquigarrow x$, we have a path $(((x, x), \mathsf{id}_x), \mathsf{id}_x) \rightsquigarrow (((a_0, a_1), q), r)$. But by induction on q and r, it suffices to show this for identity paths, namely, $(((x, x), \mathsf{id}_x), \mathsf{id}_x) \rightsquigarrow (((x, x), \mathsf{id}_x), \mathsf{id}_x)$, which is itself the identity path. □

At last we are ready to prove function extensionality, i.e., that pointwise homotopies give rise to homotopies.

`extensionality`

Theorem 3.2. *Given $f, g : A \to B$ and $h = \Pi x : A.f(x) \rightsquigarrow g(x)$, then $f \rightsquigarrow g$.*

Proof. Consider the functions $d, e : A \to Path(B)$ defined by

$$d = \lambda x.((f(x), f(x)), \mathsf{id}_{f(x)})$$
$$e = \lambda x.((f(x), g(x)), h(x)).$$

$p_0 \circ d \rightsquigarrow p_0 \circ e$ because both equal $\lambda x.f\ x$. By Propositions 3.6 and 3.7, $p_0 \circ -$ is an equivalence, so by Proposition 2.8, $d \rightsquigarrow e$. Hence $p_1 \circ d \rightsquigarrow p_1 \circ e$, which evaluates to $(\lambda x.f\ x) \rightsquigarrow (\lambda x.g\ x)$. By the eta axiom at f and g, this implies $f \rightsquigarrow g$. □

This theorem relies on univalence via the proof that post-composition by an equivalence is itself an equivalence (Proposition 3.6).

4 The interval object

The proof of Theorem 4.1, due to Mike Shulman[1], shows that the interval I as a higher inductive type, along with the eta axiom, implies function extensionality. This proof does not invoke univalence.

We posit the interval type, along with its elimination rules. Surprisingly, simple elimination and Axiom 3.2 are sufficient to derive function extensionality.

`interval_rec`

Axiom 4.1. The interval type I has elements 0 and 1 and a path $\mathsf{seg} : 0 \rightsquigarrow 1$. For any type A with $x, y : A$ and $p : x \rightsquigarrow y$, there is a function $\mathsf{rec}_I(x \overset{p}{\rightsquigarrow} y) : I \to A$ such that $\mathsf{rec}_I(p)(0) = x$, $\mathsf{rec}_I(p)(1) = y$, and $(\mathsf{rec}_I(p))_*(\mathsf{seg}) = p$.

`interval_drec`

Axiom 4.2. Given a fibration P over I, if $a : P(0)$, $b : P(1)$, and $\tau_{\mathsf{seg}}(a) \rightsquigarrow b$, then $\Pi i : I.P(i)$.

`I_extensionality`

Theorem 4.1. *Given $f, g : A \to B$ and $h = \Pi x : A.f(x) \rightsquigarrow g(x)$, then $f \rightsquigarrow g$.*

Proof. For any $x : A$, by Axiom 4.1, we have a function $\mathsf{rec}_I(h(x)) : I \to B$ satisfying the appropriate computation rules. Define $H(i) = \lambda x : A.\mathsf{rec}_I(h(x))(i)$. Notice that $H(0) = \lambda x.\mathsf{rec}_I(h(x))(0) = \lambda x.f\ x$ and $H(1) = \lambda x.g\ x$. Then $H_*(\mathsf{seg}) : (\lambda x.f\ x) \rightsquigarrow (\lambda x.g\ x)$, which by Axiom 3.2, implies $f \rightsquigarrow g$. □

[1] `http://homotopytypetheory.org/2011/04/04/an-interval-type-implies-function-extensionality/`

4.1 Factoring the univalence argument through the interval

In fact, I have noticed that the proof of Theorem 3.2 via univalence factors through the interval type in an elegant way. We must first develop more results about equivalences, starting with the converse of Proposition 2.7.

hiso_to_equiv

Proposition 4.1. *Given $f : A \to B$ and $g : B \to A$ such that $f(g(b)) \rightsquigarrow b$ and $g(f(a)) \rightsquigarrow a$, f is an equivalence.*

This proposition has been proven in various HoTT repositories.

equiv_compose

Proposition 4.2. *If $f : A \to B$ and $g : B \to C$ are equivalences, so is $g \circ f : A \to C$.*

Proof. By Proposition 4.1, it suffices to show $(g\circ f)(f^{-1}\circ g^{-1})(c) \rightsquigarrow c$ and $(f^{-1}\circ g^{-1})(g\circ f)(a) \rightsquigarrow a$, where f^{-1}, g^{-1} are witnesses of the proofs that f, g are equivalences. Then $(g \circ f)(f^{-1} \circ g^{-1})(c) = g(f(f^{-1}(g^{-1}(c)))) \rightsquigarrow g(g^{-1}(c))$ since $f(f^{-1}(b)) \rightsquigarrow b$, and similarly, $g(g^{-1}(c)) \rightsquigarrow c$. The other case follows symmetrically. □

flip_equiv

Proposition 4.3. $\mathsf{flip}(f) = \lambda x.\lambda y.f\ y\ x$ *is an equivalence* $(A \to B \to C) \to (B \to A \to C)$.

Proof. By Proposition 4.1, it suffices to show $\mathsf{flip}(\mathsf{flip}(f)) \rightsquigarrow f$ and $\mathsf{flip}(\mathsf{flip}(g)) \rightsquigarrow g$. But $\mathsf{flip}(\mathsf{flip}(f)) = \lambda x.\lambda y.(\lambda a.\lambda b.f\ b\ a)\ y\ x = \lambda x.\lambda y.f\ x\ y$, and $\lambda x.\lambda y.(f\ x)\ y \rightsquigarrow \lambda x.f\ x \rightsquigarrow f$ by Axiom 3.2. □

From the above, we conclude the analog of Proposition 3.7. Interestingly, we need function extensionality (hence, eta) at $I \to A$, as well as dependent elimination over I. A similar argument shows that $I \to A$ is equivalent to $Path(A)$.

I_src,trg

Proposition 4.4. *The projections $\hat{p}_0(f) = f(0)$, $\hat{p}_1(f) = f(1)$ are equivalences $(I \to A) \to A$.*

Proof. We consider only $\hat{p}_0$, for $\hat{p}_1$ follows symmetrically. By Proposition 4.1, we need only show that $g(a) = \lambda i : I.a$ has $\hat{p}_0(g(a)) \rightsquigarrow a$ and $g(\hat{p}_0(f)) \rightsquigarrow f$. The first is easy to see, because $\hat{p}_0(g(a)) = (\lambda i.a)(0) = a$, so id_a suffices. For the second, we must show $(\lambda i.f(0)) \rightsquigarrow f$.

Consider the fibration $P(x) = f(0) \rightsquigarrow f(x)$ over I. Then $\mathsf{id}_{f(0)} : P(0)$, $f_*(\mathsf{seg}) : P(1)$, so by Proposition 4.2, if we also have $\tau_{\mathsf{seg}}(\mathsf{id}_{f(0)}) \rightsquigarrow f_*(\mathsf{seg})$, then $\Pi x : I.P(x)$. Now consider the fibration $P'(a \overset{p}{\rightsquigarrow} b) = \Pi p' : 0 \rightsquigarrow a.\tau_p(f_*(p')) \rightsquigarrow f_*(p'p)$ over paths in I. Each fiber $P'(\mathsf{id}_a) = \Pi p' : 0 \rightsquigarrow a.\tau_{\mathsf{id}_a}(f_*(p')) \rightsquigarrow f_*(p'\mathsf{id}_a)$ is inhabited, since $\tau_{\mathsf{id}_a}(f_*(p')) = f_*(p')$, and $f_*(p'\mathsf{id}_a) \rightsquigarrow f_*(p')$. Thus by induction, we have $P'(\mathsf{seg})(\mathsf{id}_0) = \tau_{\mathsf{seg}}(f_*(\mathsf{id}_0)) \rightsquigarrow f_*(\mathsf{seg}\ \mathsf{id}_0)$, from which we obtain $\tau_{\mathsf{seg}}(\mathsf{id}_{f(0)}) \rightsquigarrow f_*(\mathsf{seg})$, which is what we needed.

Hence we conclude $\Pi x : I.f(0) \rightsquigarrow f(x)$, which we may rewrite $\Pi x : I.(\lambda i.f(0))(x) \rightsquigarrow f(x)$. Then by Theorem 4.1, we get $(\lambda i.f(0)) \rightsquigarrow f$, which completes our proof. □

I_srcflip

Proposition 4.5. $(\hat{p}_0 \circ \mathsf{flip})$ *is an equivalence* $(A \to I \to B) \to (A \to B)$.

Proof. Follows immediately from Propositions 4.4, 4.3, and 4.2. □

We can now repeat the proof of Theorem 3.2 without an appeal to univalence.

I_extensionality'

Theorem 4.2. *Given $f, g : A \to B$ and $h = \Pi x : A.f(x) \rightsquigarrow g(x)$, then $f \rightsquigarrow g$.*

Proof. Consider the functions $d, e : A \to (I \to B)$ defined by

$$\begin{aligned} d &= \lambda x.\lambda i.f(x) \\ e &= \lambda x.\mathsf{rec}_I(h(x)) \end{aligned}$$

$(\hat{p}_0 \circ \mathsf{flip})(d) \rightsquigarrow (\hat{p}_0 \circ \mathsf{flip})(e)$ because both equal $\lambda x.f\ x$. By Proposition 4.5, $(\hat{p}_0 \circ \mathsf{flip})$ is an equivalence, so by Proposition 2.8, $d \rightsquigarrow e$. Hence $(\hat{p}_1 \circ \mathsf{flip})(d) \rightsquigarrow (\hat{p}_1 \circ \mathsf{flip})(e)$, which evaluates to $(\lambda x.f\ x) \rightsquigarrow (\lambda x.g\ x)$. By the eta axiom at f and g, this implies $f \rightsquigarrow g$. $\square$

At the cost of analogy with the previous proof, we can simplify the argument by pre-flipping the arguments.

Proof. Consider the functions $d, e : I \to A \to B$ defined by

$$\begin{aligned} d &= \lambda i.\lambda x.f(x) \\ e &= \lambda i.\lambda x.\mathsf{rec}_I(h(x))\ i \end{aligned}$$

$\hat{p}_0(d) \rightsquigarrow \hat{p}_0(e)$ because both equal $\lambda x.f\ x$. By Proposition 4.4, $\hat{p}_0$ is an equivalence, so by Proposition 2.8, $d \rightsquigarrow e$. Hence $\hat{p}_1(d) \rightsquigarrow \hat{p}_1(e)$, which evaluates to $(\lambda x.f\ x) \rightsquigarrow (\lambda x.g\ x)$. By the eta axiom at f and g, this implies $f \rightsquigarrow g$. $\square$

We summarize the proofs of extensionality as follows. By univalence, $p_1 \circ -$ is an equivalence, so two A-indexed families of paths in B whose first endpoints agree for every $x : A$ are in fact homotopic; we conclude their second endpoints are homotopic as well. We visualize this proof as follows:

$$\begin{array}{cccccc} \lambda x.(& f\ x & \overset{\mathsf{id}_{f(x)}}{\rightsquigarrow} & f\ x &) \\ & \wr & \Rightarrow & \wr & \\ \lambda x.(& f\ x & \overset{h(x)}{\rightsquigarrow} & g\ x &) \end{array}$$

We are given the horizontal families of paths $\lambda x.\mathsf{id}_{f(x)}$ and $\lambda x.h(x)$; from the left vertical paths, we derive the right vertical paths, which are what we wanted to show.

On the other hand, by interval elimination, we can represent paths as maps out of I; in particular, $\lambda x.\mathsf{rec}_I(h\ x)$ is an A-indexed family of functions $I \to B$. But this is equivalent to an I-indexed family of functions $A \to B$, whose action on seg is a homotopy between those functions. We visualize this proof as follows:

$$\begin{array}{ccccc} \lambda x.(& f\ x & \overset{h(x)}{\rightsquigarrow} & g\ x &) \\ & & \Downarrow & & \\ & \lambda x.f\ x & \rightsquigarrow & \lambda x.g\ x & \end{array}$$

We are given the top family of paths, whose flipped action on seg is the bottom path.

What goes wrong if we have neither univalence nor the interval? Lacking the $I \to A$ representation of paths, we cannot use the flip trick, so we must try to repeat the univalence argument. We get stuck in the proof of Proposition 3.6, which makes essential use of equivalence induction (Theorem 3.1).

We would hope that, if f is an equivalence, then $(f \circ -)$ is an equivalence with homotopy inverse $(f^{-1} \circ -)$. It suffices to show $(f^{-1} \circ -)(f \circ -)g \rightsquigarrow g$ (and a symmetric case), which after reduction

and eta expansion, is $\lambda x.f^{-1}(f(g(x))) \rightsquigarrow \lambda x.g(x)$. We know that, for any x, $f^{-1}(f(g(x))) \rightsquigarrow g(x)$, but we cannot extend this to a homotopy between these functions—in fact, that would require precisely Theorem 3.2!

With univalence, in contrast, by Theorem 3.1 it suffices to check this fact for the identity function f. But $((\lambda a.a) \circ -)$ is an equivalence with itself as inverse, since the identity function at $g(x)$ reduces to $g(x)$, and certainly $\lambda x.g(x) \rightsquigarrow \lambda x.g(x)$.